



THE DIFFERENTIAL GEOMETRY OF PARAMETRIC PRIMITIVES

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Abstract: We derive the expressions for first and second derivatives, normal, metric matrix and curvature matrix for spheres, cones, cylinders, and tori.

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Differential Properties of Parametric Surfaces

A parametric surface is a function:

$$\mathbf{x} = \mathbf{F}(\mathbf{u})$$

where

$$\mathbf{x} = [x \ y \ z]$$

is a point in affine 3-space, and

$$\mathbf{u} = [u \ v]$$

is a point in affine 2-space.

The *Jacobian* matrix is a matrix of partial derivatives that relate changes in u and v to changes in x , y , and z :

$$\mathbf{J} = \frac{\begin{matrix} x & y & z \\ (x,y,z) \\ (u,v) \end{matrix}}{\begin{matrix} u & v \\ (u,v) \end{matrix}} = \begin{matrix} \frac{x}{u} & \frac{y}{u} & \frac{z}{u} \\ \frac{x}{v} & \frac{y}{v} & \frac{z}{v} \end{matrix} = \begin{matrix} \mathbf{x} \\ \mathbf{x} \end{matrix}$$

The Hessian is a tensor of second partial derivatives:

$$\mathbf{H} = \frac{\begin{matrix} \frac{\partial^2 x}{\partial u^2} & \frac{\partial^2 y}{\partial u^2} & \frac{\partial^2 z}{\partial u^2} \\ \frac{\partial^2 x}{\partial v u} & \frac{\partial^2 y}{\partial v u} & \frac{\partial^2 z}{\partial v u} \end{matrix}}{\begin{matrix} \frac{\partial^2 x}{\partial u^2} & \frac{\partial^2 x}{\partial u v} \\ \frac{\partial^2 x}{\partial v u} & \frac{\partial^2 x}{\partial v^2} \end{matrix}} = \begin{matrix} \frac{\partial^2 x}{\partial u^2} & \frac{\partial^2 x}{\partial u v} \\ \frac{\partial^2 x}{\partial v u} & \frac{\partial^2 x}{\partial v^2} \end{matrix}$$

The first fundamental form is defined as:

$$\mathbf{G} = \mathbf{J}\mathbf{J}^t = \begin{matrix} \frac{\mathbf{x}}{u} \cdot \frac{\mathbf{x}}{u} & \frac{\mathbf{x}}{u} \cdot \frac{\mathbf{x}}{v} \\ \frac{\mathbf{x}}{v} \cdot \frac{\mathbf{x}}{u} & \frac{\mathbf{x}}{v} \cdot \frac{\mathbf{x}}{v} \end{matrix}$$

and establishes a metric of differential length:

$$(\mathbf{dx})^2 = (\mathbf{du})\mathbf{G}(\mathbf{du})^t$$

so that the arc length of a curve segment, $\mathbf{u} = \mathbf{u}(t)$, $t_0 < t < t_1$ is given by:

$$s = \int_{t_0}^{t_1} \frac{ds}{dt} dt = \int_{t_0}^{t_1} |\dot{\mathbf{x}}| dt = \int_{t_0}^{t_1} \sqrt{(\dot{\mathbf{u}}\mathbf{G}\dot{\mathbf{u}})^t} dt$$

The differential surface area enclosed by the differential parallelogram (\mathbf{u}, \mathbf{v}) is approximately:

$$S = (\mathbf{G})^{\frac{1}{2}} \mathbf{u} \mathbf{v}$$

so that the area of a region of the surface corresponding to a region R in the $u-v$ plane is:

$$S = \int_R (\mathbf{G})^{\frac{1}{2}} du dv$$

The second fundamental matrix measures normal curvature, and is given by:

$$\mathbf{D} = \mathbf{n} \cdot \mathbf{H} = \begin{matrix} \mathbf{n} \cdot \frac{\partial^2 \mathbf{x}}{\partial u^2} & \mathbf{n} \cdot \frac{\partial^2 \mathbf{x}}{\partial u \partial v} \\ \mathbf{n} \cdot \frac{\partial^2 \mathbf{x}}{\partial v \partial u} & \mathbf{n} \cdot \frac{\partial^2 \mathbf{x}}{\partial v^2} \end{matrix}$$

The normal curvature is defined to be positive a curve \mathbf{u} on the surface turns toward the positive direction of the surface normal by:

$$\kappa_n = \frac{\dot{\mathbf{u}}\mathbf{D}\dot{\mathbf{u}}^t}{\dot{\mathbf{u}}\mathbf{G}\dot{\mathbf{u}}^t}$$

The deviation (in the normal direction) from the tangent plane of the surface, given a differential displacement of $\dot{\mathbf{u}}$ is:

$$\ddot{\mathbf{x}} \cdot \mathbf{n} = \dot{\mathbf{u}}\mathbf{D}\dot{\mathbf{u}}^t$$

Reparametrization

If the parametrization of the surface is transformed by the equations:

$$u = u(u, v) \quad \text{and} \quad v = v(u, v)$$

then the chain rule yields:

$$\frac{(x, y, z)}{(u, v)} = \frac{(u, v)}{(u, v)} \frac{(x, y, z)}{(u, v)}$$

or

$$\mathbf{J} = \mathbf{PJ}$$

where

$$\mathbf{J} = \frac{(x, y, z)}{(u, v)}$$

is the new Jacobian matrix of the surface with respect to the new parameters u and v , and

$$\mathbf{P} = \frac{(u, v)}{(u, v)} = \begin{array}{cc} \frac{u}{u} & \frac{v}{u} \\ \frac{u}{v} & \frac{v}{v} \end{array}$$

is the Jacobian matrix of the reparametrization.

The new Hessian is given by

$$\mathbf{H} = \mathbf{PHP}^T + \mathbf{QJ}$$

where

$$\mathbf{Q} = \begin{array}{cc} \frac{(u, v)}{u^2} & \frac{(u, v)}{u v} \\ \frac{(u, v)}{v u} & \frac{(u, v)}{v^2} \end{array}$$

The new fundamental matrix is given by:

$$\mathbf{G} = \mathbf{PGP}^T$$

and the new curvature matrix is given by:

$$\mathbf{D} = \mathbf{PDP}^T$$

Change of Coordinates

For simplicity, we have defined several primitives with unit size, located at the origin. Related to the reparametrization is the change of coordinates $\mathbf{x} = \mathbf{x}(\mathbf{x})$, with associated Jacobian:

$$\mathbf{C} = \frac{\mathbf{x}}{\mathbf{x}} = \begin{array}{ccc} \frac{x}{x} & \frac{y}{x} & \frac{z}{x} \\ \frac{x}{y} & \frac{y}{y} & \frac{z}{y} \\ \frac{x}{z} & \frac{y}{z} & \frac{z}{z} \end{array}$$

When the change of coordinates is represented by the affine transformation:

$$\mathbf{A} = \begin{array}{ccc} x_x & y_x & z_x \\ x_y & y_y & z_y \\ x_z & y_z & z_z \\ x_o & y_o & z_o \end{array}$$

the Jacobian is simply the submatrix:

$$\mathbf{C} = \begin{array}{ccc} x_x & y_x & z_x \\ x_y & y_y & z_y \\ x_z & y_z & z_z \end{array}$$

Regardless, the Jacobian and Hessian transform as follows:

$$\mathbf{J} = \mathbf{J}\mathbf{C}, \quad \mathbf{H} = \mathbf{H}\mathbf{C}$$

The normal is transformed as:

$$\mathbf{n} = \frac{\mathbf{n}\mathbf{C}^{-t}}{(\mathbf{n}\mathbf{C}^{-t}\mathbf{C}^{-1}\mathbf{n}^t)^{\frac{1}{2}}}$$

The denominator arises from the desire to have a *unit* normal.

The first and second fundamental matrices are then calculated as:

$$\mathbf{G} = \mathbf{J}\mathbf{J}^t = \mathbf{J}\mathbf{C}\mathbf{C}^t\mathbf{J}^t$$

$$\mathbf{D} = \mathbf{H} \cdot \mathbf{n} = \frac{(\mathbf{H}\mathbf{C}) \cdot (\mathbf{n}\mathbf{C}^{-t})}{(\mathbf{n}\mathbf{C}^{-t}\mathbf{C}^{-1}\mathbf{n}^t)^{\frac{1}{2}}} = \frac{\mathbf{H}\mathbf{C}\mathbf{C}^{-1}\mathbf{n}^t}{(\mathbf{n}\mathbf{C}^{-t}\mathbf{C}^{-1}\mathbf{n}^t)^{\frac{1}{2}}} = \frac{\mathbf{H} \cdot \mathbf{n}}{(\mathbf{n}\mathbf{C}^{-t}\mathbf{C}^{-1}\mathbf{n}^t)^{\frac{1}{2}}} = \frac{\mathbf{D}}{(\mathbf{n}\mathbf{C}^{-t}\mathbf{C}^{-1}\mathbf{n}^t)^{\frac{1}{2}}}$$

Not very pretty. But certain types of transformations can be applied easily. For a uniform scale with arbitrary translations,

$$\mathbf{C} = \begin{array}{ccc} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{array} = r\mathbf{I}$$

so that

$$\mathbf{J} = r\mathbf{J}, \quad \mathbf{H} = r\mathbf{H}, \quad \mathbf{n} = \mathbf{n}, \quad \mathbf{G} = r^2\mathbf{G}, \quad \mathbf{D} = r\mathbf{D}$$

For rotations (and arbitrary translations), the Jacobian matrix $\mathbf{C}=\mathbf{R}$ is orthogonal, so the inverse is equal to the transpose, yielding:

$$\mathbf{J} = \mathbf{J}\mathbf{R}, \quad \mathbf{H} = \mathbf{H}\mathbf{R}, \quad \mathbf{n} = \mathbf{n}\mathbf{R}, \quad \mathbf{G} = \mathbf{G}, \quad \mathbf{D} = \mathbf{D}$$

Combining the two, we have the results for a transformation that includes translations, rotations and uniform scale:

$$\mathbf{J} = r\mathbf{J}\mathbf{R}, \quad \mathbf{H} = r\mathbf{H}\mathbf{R}, \quad \mathbf{n} = \mathbf{n}\mathbf{R}, \quad \mathbf{G} = r^2\mathbf{G}, \quad \mathbf{D} = r\mathbf{D}$$

or in terms of the composite matrix $\mathbf{C} = r\mathbf{R}$:

$$\mathbf{J} = \mathbf{J}\mathbf{C}, \quad \mathbf{H} = \mathbf{H}\mathbf{C}, \quad \mathbf{n} = \frac{\mathbf{n}\mathbf{C}}{(|\mathbf{C}|)}, \quad \mathbf{G} = (|\mathbf{C}|)^{\frac{2}{3}}\mathbf{G}, \quad \mathbf{D} = (|\mathbf{C}|)^{\frac{1}{3}}\mathbf{D}$$

Sphere

Given the spherical coordinates:

$$[x \ y \ z] = [r \sin \theta \cos \phi \quad r \sin \theta \sin \phi \quad r \cos \theta]$$

we have the Jacobian matrix:

$$\frac{(x,y,z)}{(r, \theta)} = \begin{bmatrix} -y & x & 0 \\ xz & yz & -\sqrt{x^2+y^2} \\ \sqrt{x^2+y^2} & \sqrt{x^2+y^2} & 0 \end{bmatrix}$$

the Hessian tensor:

$$\frac{^2(x,y,z)}{(r, \theta)(r, \theta)} = \begin{bmatrix} [-x & -y & 0] & -\frac{yz}{\sqrt{x^2+y^2}} & \frac{xz}{\sqrt{x^2+y^2}} & 0 \\ -\frac{yz}{\sqrt{x^2+y^2}} & \frac{xz}{\sqrt{x^2+y^2}} & 0 & [-x & -y & -z] \end{bmatrix}$$

the first fundamental form:

$$\mathbf{G} = \begin{bmatrix} x^2+y^2 & 0 \\ 0 & r^2 \end{bmatrix}$$

the normal:

$$\mathbf{n} = \frac{x}{r} \frac{y}{r} \frac{z}{r}$$

and the second fundamental form:

$$\mathbf{D} = \begin{bmatrix} -\frac{x^2+y^2}{r} & 0 \\ 0 & -r \end{bmatrix}$$

Unit Sphere

Angle Parametrization

Given the unit spherical coordinates with $0 < \theta < 2\pi$, $0 < \phi < \pi$, we parametrize the sphere:

$$[x \ y \ z] = [\sin \phi \cos \theta \ \sin \phi \sin \theta \ \cos \phi]$$

This yields the Jacobian matrix:

$$\mathbf{J} = \begin{bmatrix} -y & x & 0 \\ \frac{xz}{\sqrt{x^2+y^2}} & \frac{yz}{\sqrt{x^2+y^2}} & -\sqrt{x^2+y^2} \end{bmatrix}$$

the Hessian tensor:

$$\mathbf{H} = \begin{bmatrix} [-x \ -y \ 0] & -\frac{yz}{\sqrt{x^2+y^2}} & \frac{xz}{\sqrt{x^2+y^2}} & 0 \\ -\frac{yz}{\sqrt{x^2+y^2}} & \frac{xz}{\sqrt{x^2+y^2}} & 0 & [-x \ -y \ -z] \end{bmatrix}$$

the first fundamental form:

$$\mathbf{G} = \begin{bmatrix} x^2+y^2 & 0 \\ 0 & 1 \end{bmatrix}$$

the normal:

$$\mathbf{n} = [x \ y \ z]$$

and the second fundamental form:

$$\mathbf{D} = \begin{bmatrix} -(x^2+y^2) & 0 \\ 0 & -1 \end{bmatrix}$$

Angle Parametrization

With the reparametrization $\theta = 2u$, $\phi = v$, we have the Jacobian:

$$\mathbf{P} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Applying the chain rule, we have:

$$\mathbf{J}_{uv} = \begin{bmatrix} -2y & 2x & 0 \\ \frac{xz}{\sqrt{x^2+y^2}} & \frac{yz}{\sqrt{x^2+y^2}} & -\sqrt{x^2+y^2} \end{bmatrix}$$

$$\mathbf{H}_{uv} = \begin{bmatrix} 4x & -2y & 0 \\ -\frac{yz}{\sqrt{x^2+y^2}} & \frac{xz}{\sqrt{x^2+y^2}} & 0 \\ 2x & -2y & -2z \end{bmatrix}$$

$$\mathbf{G}_{uv} = \begin{bmatrix} 4(x^2+y^2) & 0 \\ 0 & 4z^2 \end{bmatrix}$$

$$\mathbf{D}_{uv} = \begin{bmatrix} -4x & 0 \\ 0 & -2z \end{bmatrix}$$

Changing coordinates to yield a sphere of arbitrary radius, we find that the expressions for the Jacobian, the Hessian, and the metric matrix remain the same, because x , y , and z scale linearly with r . The curvature matrix changes to:

$$\mathbf{D}_{uv} = \begin{bmatrix} -\frac{4x}{r} & 0 \\ 0 & -\frac{2z}{r} \end{bmatrix}$$

Cone

Angle Parametrization

Given the unit conical parametrization:

$$[x \ y \ z] = [z \cos \quad z \sin \quad z]$$

we have the Jacobian matrix:

$$\mathbf{J}_z = \begin{bmatrix} -y & x & 0 \\ \frac{x}{z} & \frac{y}{z} & 1 \\ z & z & z \end{bmatrix}$$

the Hessian tensor:

$$\mathbf{H}_z = \begin{bmatrix} [-x & -y & 0] & -\frac{y}{z} & \frac{x}{z} & 0 \\ -\frac{y}{z} & \frac{x}{z} & 0 & [0 & 0 & 0] \end{bmatrix}$$

the first fundamental form:

$$\mathbf{G}_z = \begin{bmatrix} x^2 + y^2 & 0 \\ 0 & \frac{x^2 + y^2 + z^2}{z^2} \end{bmatrix} = \begin{bmatrix} z^2 & 0 \\ 0 & 2 \end{bmatrix}$$

the normal:

$$\mathbf{n}_z = \frac{x}{z\sqrt{2}} \quad \frac{y}{z\sqrt{2}} \quad -\frac{1}{\sqrt{2}}$$

and the second fundamental form:

$$\mathbf{D}_z = \begin{bmatrix} -\frac{z}{\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix}$$

Unit Parametrization

For the parametrization:

$$[x \ y \ z] = [r \cos 2 \ u \ r \sin 2 \ u \ v \ h]$$

we have:

$$\mathbf{J}_{uv} = \begin{bmatrix} -2 \ y & 2 \ x & 0 \\ \frac{hx}{rz} & \frac{hy}{rz} & h \end{bmatrix}$$

$$\mathbf{H}_{uv} = \begin{bmatrix} 4 \ ^2[-x & -y & 0] & \frac{2 \ h}{rz}[-y & x & 0] \\ \frac{2 \ h}{rz}[-y & x & 0] & [0 & 0 & 0] \end{bmatrix}$$

$$\mathbf{G}_{uv} = \begin{pmatrix} 4(x^2 + y^2) & 0 \\ 0 & \frac{h^2(x^2 + y^2 + z^2)}{z^2} \end{pmatrix}$$

$$\mathbf{n}_{uv} = \frac{1}{\sqrt{1+h^2}} \begin{pmatrix} \frac{h^2x}{rz} & \frac{h^2y}{rz} & -1 \end{pmatrix}$$

$$\mathbf{D}_{uv} = \begin{pmatrix} -\frac{4x^2yz}{\sqrt{1+h^2}} & 0 \\ 0 & 0 \end{pmatrix}$$

Cylinder

Angle Parametrization

Given the cylindrical parametrization:

$$[x \ y \ z] = [\cos \ \sin \ z]$$

we have the Jacobian matrix:

$$\mathbf{J} = \begin{bmatrix} -y & x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

the Hessian tensor:

$$\mathbf{H} = \begin{bmatrix} -x & -y & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

the first fundamental form:

$$\mathbf{G} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

the normal:

$$\mathbf{n} = [x \ y \ 0]$$

and the second fundamental form:

$$\mathbf{D} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

Unit Parametrization

With the parametrization:

$$[x \ y \ z] = [r \cos 2u \ r \sin 2u \ hv]$$

we have the Jacobian matrix:

$$\mathbf{J}_{uv} = \begin{bmatrix} -2y & 2x & 0 \\ 0 & 0 & h \end{bmatrix}$$

the Hessian tensor:

$$\mathbf{H}_{uv} = \begin{bmatrix} -4x & -4y & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

the first fundamental form:

$$\mathbf{G}_{uv} = \begin{bmatrix} 4r^2 & 0 \\ 0 & h^2 \end{bmatrix}$$

the normal:

$$\mathbf{n} = \frac{x}{r} \frac{y}{r} 0$$

and the second fundamental form:

$$\mathbf{D}_{uv} = \begin{matrix} -4 & 2r & 0 \\ 0 & 0 & 0 \end{matrix}$$

Torus

Angle Parametrization

Given the torus parametrization:

$$[x \ y \ z] = [(R + r \cos \theta) \cos \phi \quad (R + r \cos \theta) \sin \phi \quad r \sin \theta]$$

we have the Jacobian matrix:

$$\mathbf{J} = \begin{bmatrix} -y & x & 0 \\ -\frac{xz}{\sqrt{x^2 + y^2}} & -\frac{yz}{\sqrt{x^2 + y^2}} & \sqrt{x^2 + y^2} - R \end{bmatrix}$$

the Hessian tensor:

$$\mathbf{H} = \begin{bmatrix} [-x \ -y \ 0] & \frac{yz}{\sqrt{x^2 + y^2}} & -\frac{xz}{\sqrt{x^2 + y^2}} & 0 \\ \frac{yz}{\sqrt{x^2 + y^2}} & -\frac{xz}{\sqrt{x^2 + y^2}} & 0 & -x \left(1 - \frac{R}{\sqrt{x^2 + y^2}}\right) \\ \frac{yz}{\sqrt{x^2 + y^2}} & -\frac{xz}{\sqrt{x^2 + y^2}} & 0 & -y \left(1 - \frac{R}{\sqrt{x^2 + y^2}}\right) \\ 0 & -x \left(1 - \frac{R}{\sqrt{x^2 + y^2}}\right) & -y \left(1 - \frac{R}{\sqrt{x^2 + y^2}}\right) & -z \end{bmatrix}$$

the first fundamental form:

$$\mathbf{G} = \begin{bmatrix} x^2 + y^2 & 0 \\ 0 & r^2 \end{bmatrix}$$

the normal:

$$\mathbf{n} = \begin{bmatrix} x \frac{1 - \frac{R}{\sqrt{x^2 + y^2}}}{r} & y \frac{1 - \frac{R}{\sqrt{x^2 + y^2}}}{r} & \frac{z}{r} \end{bmatrix}$$

and the second fundamental form:

$$\mathbf{D} = \begin{bmatrix} -\frac{x^2 + y^2}{r} & 1 - \frac{R}{\sqrt{x^2 + y^2}} & 0 \\ 0 & 0 & -r \end{bmatrix}$$

$$= \begin{bmatrix} \frac{R^2 - x^2 - y^2 + z^2 - r^2}{2r} & 0 \\ 0 & -r \end{bmatrix}$$

using the torus's implicit equation:

$$\left(\sqrt{x^2 + y^2} - R\right)^2 + z^2 = r^2$$