Computing 2D Polygon Moments Using Green's Theorem

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Green's Theorem

In this paper, we derive the expressions for zero-, first-, and second-order moments for two-dimensional polygons, using Green's Theorem to convert an area integral to a line integral, then evaluating this in terms of the polygon vertex coordinates. Green's Theoren is given as:

ContourIntegral[M ⅆx + N ⅆy] = AreaIntegral[(∂_x N - ∂_y M) ⅆx ⅆy]

Parametric Definitions, for Contour Integrals

- **Edge**

It is useful to define polygon edges in terms of a parameter that ranges from zero to one. This then simplifies the evaluation of the line integrals.

```
Ex = x0 + t (x1 - x0)
```

```
x0 + t (-x0 + x1)
```

```
Ey = y0 + t (y1 - y0)
```

$y0 + t\,(-y0 + y1)$

- **Differentials**

The differentials dx and dy are then related to dt by:

```
dx = (x1 - x0) dt
```

$(-x0 + x1)$ DifferentialD[t]

```
dy = (y1 - y0) dt
```

$(-y0 + y1)$ DifferentialD[t]

Area (Zeroth Moment)

The zeroth moment of a polygon is its area.

$$A = \int\int dx\,dy$$

- **Green's Function #1**

$$A = \int\int dx\,dy = \frac{1}{2}\left(-\int y\,dx + \int x\,dy\right)$$

Evaluating the line integral along the edge of the polygon from (x0,y0) to (x1,y1), we have:

$$\mathbf{AE1} = \frac{1}{2}\int_0^1 (-\mathbf{Ey}\,(x1 - x0) + \mathbf{Ex}\,(y1 - y0))\,dt$$

$$\frac{1}{2}\,(-x1\,y0 + x0\,y1)$$

The zeroth moment is then the sum of the expression evauated at each edge.

This is the well-known cross product or parallelogram rule.

- **Green's Function #2**

$$A = \int\int dx\, dy = -\int y\, dx$$

Evaluating the line integral along the edge of the polygon from (x0,y0) to (x1,y1), we have:

$$\mathbf{AE2 = -Simplify}\left[\int_0^1 \mathbf{Ey\ (x1 - x0)}\, dt\right]$$

$$\frac{1}{2}\ (x0 - x1)\ (y0 + y1)$$

This is the trapezoidal rule.

- **Green's Function #3**

$$A = \int\int dx\, dy = \int x\, dy$$

Evaluating the line integral along the edge of the polygon from (x0,y0) to (x1,y1), we have:

$$\mathbf{AE3 = Simplify}\left[\int_0^1 \mathbf{Ex\ (y1 - y0)}\, dt\right]$$

$$-\frac{1}{2}\ (x0 + x1)\ (y0 - y1)$$

Yet another variation of the trapezoidal rule.

First Moments

The first moment of a polygon is its centroid.

$$Cx = \int\int x\, dx\, dy$$

$$Cy = \int\int y\, dx\, dy$$

- **Green's Functions #1**

$$Cx = \int\int x\, dx\, dy = \frac{1}{2}\int x^2\, dy$$

$$Cy = \int\int y\, dx\, dy = -\frac{1}{2}\int y^2\, dx$$

Evaluating the line integral along the edge of the polygon from (x0,y0) to (x1,y1), we have:

```
CEx = Simplify[1/2 ∫_0^1 Ex^2 (y1 - y0) dt]
```

$$-\frac{1}{6}\left(x0^2 + x0\, x1 + x1^2\right)(y0 - y1)$$

```
CEy = Simplify[-1/2 ∫_0^1 Ey^2 (x1 - x0) dt]
```

$$\frac{1}{6}(x0 - x1)\left(y0^2 + y0\, y1 + y1^2\right)$$

Unit Square Test

```
(CEx /. {x0 -> 0, x1 -> 1, y0 -> 0, y1 -> 0}) +
(CEx /. {x0 -> 1, x1 -> 1, y0 -> 0, y1 -> 1}) +
(CEx /. {x0 -> 1, x1 -> 0, y0 -> 1, y1 -> 1}) +
(CEx /. {x0 -> 0, x1 -> 0, y0 -> 1, y1 -> 0})
```

$$\frac{1}{2}$$

```
(CEy /. {x0 -> 0, x1 -> 1, y0 -> 0, y1 -> 0}) +
(CEy /. {x0 -> 1, x1 -> 1, y0 -> 0, y1 -> 1}) +
(CEy /. {x0 -> 1, x1 -> 0, y0 -> 1, y1 -> 1}) +
(CEy /. {x0 -> 0, x1 -> 0, y0 -> 1, y1 -> 0})
```

$$\frac{1}{2}$$

Generic Quadrilateral Test

```
(CEx /. {x0 -> a0, x1 -> a1, y0 -> b0, y1 -> b1}) +
(CEx /. {x0 -> a1, x1 -> a2, y0 -> b1, y1 -> b2}) +
(CEx /. {x0 -> a2, x1 -> a3, y0 -> b2, y1 -> b3}) +
(CEx /. {x0 -> a3, x1 -> a0, y0 -> b3, y1 -> b0})
```

$$-\frac{1}{6}\left(a0^2 + a0\, a1 + a1^2\right)(b0 - b1) - \frac{1}{6}\left(a1^2 + a1\, a2 + a2^2\right)(b1 - b2) - \frac{1}{6}\left(a2^2 + a2\, a3 + a3^2\right)(b2 - b3) - \frac{1}{6}\left(a0^2 + a0\, a3 + a3^2\right)(-b0 + b3)$$

Generic Rectangle Test

```
Simplify[
(CEx /. {x0 -> a0, x1 -> a1, y0 -> b0, y1 -> b0}) +
(CEx /. {x0 -> a1, x1 -> a1, y0 -> b0, y1 -> b1}) +
(CEx /. {x0 -> a1, x1 -> a0, y0 -> b1, y1 -> b1}) +
(CEx /. {x0 -> a0, x1 -> a0, y0 -> b1, y1 -> b0})
]
```

$$\frac{1}{2}\left(a0^2 - a1^2\right)(b0 - b1)$$

■ Greens Functions #2

$$Cx = \int\int x \, dx \, dy = -\int x \, y \, dx$$

$$Cy = \int\int y \, dx \, dy = \int x \, y \, dy$$

Evaluating the line integral along the edge of the polygon from (x0,y0) to (x1,y1), we have:

$$\mathbf{CEx1 = Simplify}\left[-\int_0^1 \mathbf{Ex\, Ey\, (x1 - x0)}\, dt\right]$$

$$\frac{1}{6}(x0 - x1)(x0(2\,y0 + y1) + x1(y0 + 2\,y1))$$

$$\mathbf{CEy1 = Simplify}\left[\int_0^1 \mathbf{Ex\, Ey\, (y1 - y0)}\, dt\right]$$

$$-\frac{1}{6}(y0 - y1)(x0(2\,y0 + y1) + x1(y0 + 2\,y1))$$

Unit Square Test

```
(CEx1 /. {x0 -> 0, x1 -> 1, y0 -> 0, y1 -> 0}) +
(CEx1 /. {x0 -> 1, x1 -> 1, y0 -> 0, y1 -> 1}) +
(CEx1 /. {x0 -> 1, x1 -> 0, y0 -> 1, y1 -> 1}) +
(CEx1 /. {x0 -> 0, x1 -> 0, y0 -> 1, y1 -> 0})
```

$$\frac{1}{2}$$

```
(CEy1 /. {x0 -> 0, x1 -> 1, y0 -> 0, y1 -> 0}) +
(CEy1 /. {x0 -> 1, x1 -> 1, y0 -> 0, y1 -> 1}) +
(CEy1 /. {x0 -> 1, x1 -> 0, y0 -> 1, y1 -> 1}) +
(CEy1 /. {x0 -> 0, x1 -> 0, y0 -> 1, y1 -> 0})
```

$$\frac{1}{2}$$

Second Moments

$$\text{Ixx} = \int\int x^2 \, dx \, dy$$

$$\text{Ixy} = \int\int x \, y \, dx \, dy$$

$$\text{Iyy} = \int\int y^2 \, dx \, dy$$

- **Greens Functions #1**

$$\text{Ixx} = \int\int x^2 \, dx \, dy = \frac{1}{3} \int x^3 \, dy$$

$$\text{Ixy} = \int\int xy \, dx \, dy = \frac{1}{4} \left(-\int x \, y^2 \, dx + \int x^2 \, y \, dy\right)$$

$$\text{Iyy} = \int\int y^2 \, dx \, dy = -\frac{1}{3} \int y^3 \, dx$$

Evaluating the line integral along the edge of the polygon from (x0,y0) to (x1,y1), we have:

$$\textbf{IExx = Simplify}\left[\frac{1}{3} \int_0^1 \text{Ex}^3 \, (\text{y1} - \text{y0}) \, dt\right]$$

$$-\frac{1}{12} \left(\text{x0}^3 + \text{x0}^2 \, \text{x1} + \text{x0} \, \text{x1}^2 + \text{x1}^3\right) (\text{y0} - \text{y1})$$

$$\textbf{IExy = Simplify}\left[\frac{1}{4} \left(\int_0^1 \left(-\text{Ex} \, \text{Ey}^2 \, (\text{x1} - \text{x0}) + \text{Ex}^2 \, \text{Ey} \, (\text{y1} - \text{y0})\right) dt\right)\right]$$

$$\frac{1}{24} \left(\text{x0}^2 \, \text{y1} \, (2 \, \text{y0} + \text{y1}) - \text{x1}^2 \, \text{y0} \, (\text{y0} + 2 \, \text{y1}) + 2 \, \text{x0} \, \text{x1} \, (-\text{y0}^2 + \text{y1}^2)\right)$$

$$\mathbf{IEyy = Simplify\left[-\frac{1}{3}\int_0^1 Ey^3\ (x1\ -x0)\ dt\right]}$$

$$\frac{1}{12}\ (x0 - x1)\ (y0^3 + y0^2\ y1 + y0\ y1^2 + y1^3)$$

Unit Square Test

$$\left(\text{IExx /. }\left\{x0\to-\frac{1}{2},\ x1\to\frac{1}{2},\ y0\to-\frac{1}{2},\ y1\to-\frac{1}{2}\right\}\right)+\left(\text{IExx /. }\left\{x0\to\frac{1}{2},\ x1\to\frac{1}{2},\ y0\to-\frac{1}{2},\ y1\to\frac{1}{2}\right\}\right)+\left(\text{IExx /. }\left\{x0\to\frac{1}{2},\ x1\to-\frac{1}{2},\ y0\to\frac{1}{2},\ y1\to\frac{1}{2}\right\}\right)+\left(\text{IExx /. }\left\{x0\to-\frac{1}{2},\ x1\to-\frac{1}{2},\ y0\to\frac{1}{2},\ y1\to-\frac{1}{2}\right\}\right)$$

$$\frac{1}{12}$$

$$\left(\text{IExy /. }\left\{x0\to-\frac{1}{2},\ x1\to\frac{1}{2},\ y0\to-\frac{1}{2},\ y1\to-\frac{1}{2}\right\}\right)+\left(\text{IExy /. }\left\{x0\to\frac{1}{2},\ x1\to\frac{1}{2},\ y0\to-\frac{1}{2},\ y1\to\frac{1}{2}\right\}\right)+\left(\text{IExy /. }\left\{x0\to\frac{1}{2},\ x1\to-\frac{1}{2},\ y0\to\frac{1}{2},\ y1\to\frac{1}{2}\right\}\right)+\left(\text{IExy /. }\left\{x0\to-\frac{1}{2},\ x1\to-\frac{1}{2},\ y0\to\frac{1}{2},\ y1\to-\frac{1}{2}\right\}\right)$$

0

$$\left(\text{IEyy /. }\left\{x0\to-\frac{1}{2},\ x1\to\frac{1}{2},\ y0\to-\frac{1}{2},\ y1\to-\frac{1}{2}\right\}\right)+\left(\text{IEyy /. }\left\{x0\to\frac{1}{2},\ x1\to\frac{1}{2},\ y0\to-\frac{1}{2},\ y1\to\frac{1}{2}\right\}\right)+\left(\text{IEyy /. }\left\{x0\to\frac{1}{2},\ x1\to-\frac{1}{2},\ y0\to\frac{1}{2},\ y1\to\frac{1}{2}\right\}\right)+\left(\text{IEyy /. }\left\{x0\to-\frac{1}{2},\ x1\to-\frac{1}{2},\ y0\to\frac{1}{2},\ y1\to-\frac{1}{2}\right\}\right)$$

$$\frac{1}{12}$$

Generic Rectangle Test

```
Simplify[
    (IExx /. {x0 -> a0, x1 -> a1, y0 -> b0, y1 -> b0}) +
    (IExx /. {x0 -> a1, x1 -> a1, y0 -> b0, y1 -> b1}) +
    (IExx /. {x0 -> a1, x1 -> a0, y0 -> b1, y1 -> b1}) +
    (IExx /. {x0 -> a0, x1 -> a0, y0 -> b1, y1 -> b0})
    ]
```

$\frac{1}{3}\left(a0^3 - a1^3\right)(b0 - b1)$

```
Simplify[
    (IExy /. {x0 -> a0, x1 -> a1, y0 -> b0, y1 -> b0}) +
    (IExy /. {x0 -> a1, x1 -> a1, y0 -> b0, y1 -> b1}) +
    (IExy /. {x0 -> a1, x1 -> a0, y0 -> b1, y1 -> b1}) +
    (IExy /. {x0 -> a0, x1 -> a0, y0 -> b1, y1 -> b0})
    ]
```

$\frac{1}{4}\left(a0^2 - a1^2\right)\left(b0^2 - b1^2\right)$

```
Simplify[
    (IEyy /. {x0 -> a0, x1 -> a1, y0 -> b0, y1 -> b0}) +
    (IEyy /. {x0 -> a1, x1 -> a1, y0 -> b0, y1 -> b1}) +
    (IEyy /. {x0 -> a1, x1 -> a0, y0 -> b1, y1 -> b1}) +
    (IEyy /. {x0 -> a0, x1 -> a0, y0 -> b1, y1 -> b0})
    ]
```

$\frac{1}{3}(a0 - a1)\left(b0^3 - b1^3\right)$

Centered Rectangle Test

$$\begin{aligned}&\textbf{Simplify}\Big[\\
&\quad\left(\textbf{IExx} \;/.\; \left\{\textbf{x0} \to -\frac{\textbf{w}}{2},\ \textbf{x1} \to \frac{\textbf{w}}{2},\ \textbf{y0} \to -\frac{\textbf{h}}{2},\ \textbf{y1} \to -\frac{\textbf{h}}{2}\right\}\right) +\\
&\quad\left(\textbf{IExx} \;/.\; \left\{\textbf{x0} \to \frac{\textbf{w}}{2},\ \textbf{x1} \to \frac{\textbf{w}}{2},\ \textbf{y0} \to -\frac{\textbf{h}}{2},\ \textbf{y1} \to \frac{\textbf{h}}{2}\right\}\right) +\\
&\quad\left(\textbf{IExx} \;/.\; \left\{\textbf{x0} \to \frac{\textbf{w}}{2},\ \textbf{x1} \to -\frac{\textbf{w}}{2},\ \textbf{y0} \to \frac{\textbf{h}}{2},\ \textbf{y1} \to \frac{\textbf{h}}{2}\right\}\right) +\\
&\quad\left(\textbf{IExx} \;/.\; \left\{\textbf{x0} \to -\frac{\textbf{w}}{2},\ \textbf{x1} \to -\frac{\textbf{w}}{2},\ \textbf{y0} \to \frac{\textbf{h}}{2},\ \textbf{y1} \to -\frac{\textbf{h}}{2}\right\}\right)\\
&\quad\Big]\end{aligned}$$

$\frac{h\,w^3}{12}$

```
Simplify[
  (IExy /. {x0 -> -w/2, x1 -> w/2, y0 -> -h/2, y1 -> -h/2}) +
  (IExy /. {x0 -> w/2, x1 -> w/2, y0 -> -h/2, y1 -> h/2}) +
  (IExy /. {x0 -> w/2, x1 -> -w/2, y0 -> h/2, y1 -> h/2}) +
  (IExy /. {x0 -> -w/2, x1 -> -w/2, y0 -> h/2, y1 -> -h/2})
  ]
```

0

```
Simplify[
  (IEyy /. {x0 -> -w/2, x1 -> w/2, y0 -> -h/2, y1 -> -h/2}) +
  (IEyy /. {x0 -> w/2, x1 -> w/2, y0 -> -h/2, y1 -> h/2}) +
  (IEyy /. {x0 -> w/2, x1 -> -w/2, y0 -> h/2, y1 -> h/2}) +
  (IEyy /. {x0 -> -w/2, x1 -> -w/2, y0 -> h/2, y1 -> -h/2})
  ]
```

$$\frac{h^3\,w}{12}$$

- **Green's Functions #2**

$$Ixx = \int\int x^2\,dx\,dy = -\int x^2\,y\,dx$$

$$Ixy = \int\int xy\,dx\,dy = -\int xy^2\,dx$$

$$Iyy = \int\int y^2\,dx\,dy = \int xy^2\,dy$$

Evaluating the line integral along the edge of the polygon from (x0,y0) to (x1,y1), we have:

$$\textbf{IExx2} = \textbf{Simplify}\left[-\int_0^1 \textbf{Ex}^2\ \textbf{Ey}\ (\textbf{x1} - \textbf{x0})\ dt\right]$$

$$\frac{1}{12}\ (x0 - x1)\ (2\ x0\ x1\ (y0 + y1) + x0^2\ (3\ y0 + y1) + x1^2\ (y0 + 3\ y1))$$

$$\textbf{IExy2} = \textbf{Simplify}\left[-\frac{1}{2}\left(\int_0^1 (\textbf{Ex}\ \textbf{Ey}^2\ (\textbf{x1} - \textbf{x0}))\ dt\right)\right]$$

$$\frac{1}{24}\ (2\ x0\ x1\ (-y0^2 + y1^2) + x0^2\ (3\ y0^2 + 2\ y0\ y1 + y1^2) - x1^2\ (y0^2 + 2\ y0\ y1 + 3\ y1^2))$$

$$\textbf{IEyy2} = \textbf{Simplify}\left[\int_0^1 \textbf{Ex}\ \textbf{Ey}^2\ (\textbf{y1} - \textbf{y0})\ dt\right]$$

$$-\frac{1}{12}\ (y0 - y1)\ (x0\ (3\ y0^2 + 2\ y0\ y1 + y1^2) + x1\ (y0^2 + 2\ y0\ y1 + 3\ y1^2))$$

Unit Square Test

$$\left(\textbf{IExx2 /. }\left\{x0 \to -\frac{1}{2},\ x1 \to \frac{1}{2},\ y0 \to -\frac{1}{2},\ y1 \to -\frac{1}{2}\right\}\right) + \left(\textbf{IExx2 /. }\left\{x0 \to \frac{1}{2},\ x1 \to \frac{1}{2},\ y0 \to -\frac{1}{2},\ y1 \to \frac{1}{2}\right\}\right) + \left(\textbf{IExx2 /. }\left\{x0 \to \frac{1}{2},\ x1 \to -\frac{1}{2},\ y0 \to \frac{1}{2},\ y1 \to \frac{1}{2}\right\}\right) + \left(\textbf{IExx2 /. }\left\{x0 \to -\frac{1}{2},\ x1 \to -\frac{1}{2},\ y0 \to \frac{1}{2},\ y1 \to -\frac{1}{2}\right\}\right)$$

$$\frac{1}{12}$$

$$\left(\textbf{IExy2 /. }\left\{x0 \to -\frac{1}{2},\ x1 \to \frac{1}{2},\ y0 \to -\frac{1}{2},\ y1 \to -\frac{1}{2}\right\}\right) + \left(\textbf{IExy2 /. }\left\{x0 \to \frac{1}{2},\ x1 \to \frac{1}{2},\ y0 \to -\frac{1}{2},\ y1 \to \frac{1}{2}\right\}\right) + \left(\textbf{IExy2 /. }\left\{x0 \to \frac{1}{2},\ x1 \to -\frac{1}{2},\ y0 \to \frac{1}{2},\ y1 \to \frac{1}{2}\right\}\right) + \left(\textbf{IExy2 /. }\left\{x0 \to -\frac{1}{2},\ x1 \to -\frac{1}{2},\ y0 \to \frac{1}{2},\ y1 \to -\frac{1}{2}\right\}\right)$$

$$0$$

$$\left(\text{IEyy2 /. }\{x0 \to -\tfrac{1}{2},\ x1 \to \tfrac{1}{2},\ y0 \to -\tfrac{1}{2},\ y1 \to -\tfrac{1}{2}\}\right) +$$
$$\left(\text{IEyy2 /. }\{x0 \to \tfrac{1}{2},\ x1 \to \tfrac{1}{2},\ y0 \to -\tfrac{1}{2},\ y1 \to \tfrac{1}{2}\}\right) +$$
$$\left(\text{IEyy2 /. }\{x0 \to \tfrac{1}{2},\ x1 \to -\tfrac{1}{2},\ y0 \to \tfrac{1}{2},\ y1 \to \tfrac{1}{2}\}\right) +$$
$$\left(\text{IEyy2 /. }\{x0 \to -\tfrac{1}{2},\ x1 \to -\tfrac{1}{2},\ y0 \to \tfrac{1}{2},\ y1 \to -\tfrac{1}{2}\}\right)$$

$$\frac{1}{12}$$

- **Green's Functions #3**

$$\text{Ixy} = \int\int xy\, dx\, dy = \int x^2\, y\, dy$$

Evaluating the line integral along the edge of the polygon from (x0,y0) to (x1,y1), we have:

$$\text{IExy3 = Simplify}\left[\frac{1}{2}\left(\int_0^1 (\text{Ex}^2\ \text{Ey}\ (y1 - y0))\, dt\right)\right]$$

$$-\frac{1}{24}\ (y0 - y1)\ (2\ x0\ x1\ (y0 + y1) + x0^2\ (3\ y0 + y1) + x1^2\ (y0 + 3\ y1))$$

Unit Square Test

$$\left(\text{IExy3 /. }\{x0 \to -\tfrac{1}{2},\ x1 \to \tfrac{1}{2},\ y0 \to -\tfrac{1}{2},\ y1 \to -\tfrac{1}{2}\}\right) +$$
$$\left(\text{IExy3 /. }\{x0 \to \tfrac{1}{2},\ x1 \to \tfrac{1}{2},\ y0 \to -\tfrac{1}{2},\ y1 \to \tfrac{1}{2}\}\right) +$$
$$\left(\text{IExy3 /. }\{x0 \to \tfrac{1}{2},\ x1 \to -\tfrac{1}{2},\ y0 \to \tfrac{1}{2},\ y1 \to \tfrac{1}{2}\}\right) +$$
$$\left(\text{IExy3 /. }\{x0 \to -\tfrac{1}{2},\ x1 \to -\tfrac{1}{2},\ y0 \to \tfrac{1}{2},\ y1 \to -\tfrac{1}{2}\}\right)$$

0

Alternate Definitions in terms of average and half difference

```
Simplify[AE1 /. {x0 -> xs - xd, x1 -> xs + xd, y0 -> ys - yd, y1 -> ys + yd}]
```

xs yd − xd ys

```
Simplify[AE2 /. {x0 -> xs - xd, x1 -> xs + xd, y0 -> ys - yd, y1 -> ys + yd}]
```

$-2\, xd\, ys$

```
Simplify[AE3 /. {x0 -> xs - xd, x1 -> xs + xd, y0 -> ys - yd, y1 -> ys + yd}]
```

$2\, xs\, yd$

```
Simplify[CEx /. {x0 -> xs - xd, x1 -> xs + xd, y0 -> ys - yd, y1 -> ys + yd}]
```

$\frac{1}{3}\, (xd^2 + 3\, xs^2)\, yd$

```
Simplify[CEy /. {x0 -> xs - xd, x1 -> xs + xd, y0 -> ys - yd, y1 -> ys + yd}]
```

$-\frac{1}{3}\, xd\, (yd^2 + 3\, ys^2)$

```
Simplify[
 CEx1 /. {x0 -> xs - xd, x1 -> xs + xd, y0 -> ys - yd, y1 -> ys + yd}]
```

$-\frac{2}{3}\, xd\, (xd\, yd + 3\, xs\, ys)$

```
Simplify[
 CEx1 /. {x0 -> xs - xd, x1 -> xs + xd, y0 -> ys - yd, y1 -> ys + yd}]
```

$-\frac{2}{3}\, xd\, (xd\, yd + 3\, xs\, ys)$

```
Simplify[
 IExx /. {x0 -> xs - xd, x1 -> xs + xd, y0 -> ys - yd, y1 -> ys + yd}]
```

$\frac{2}{3}\, xs\, (xd^2 + xs^2)\, yd$

```
Simplify[
 IExy /. {x0 -> xs - xd, x1 -> xs + xd, y0 -> ys - yd, y1 -> ys + yd}]
```

$\frac{1}{6}\, (-xd^2\, yd\, ys + 3\, xs^2\, yd\, ys + xd\, xs\, (yd^2 - 3\, ys^2))$

```
Simplify[
 IEyy /. {x0 -> xs - xd, x1 -> xs + xd, y0 -> ys - yd, y1 -> ys + yd}]
```

$-\frac{2}{3}\, xd\, ys\, (yd^2 + ys^2)$

```
Simplify[
 IExx2 /. {x0 -> xs - xd, x1 -> xs + xd, y0 -> ys - yd, y1 -> ys + yd}]
```

$-\frac{2}{3}\, xd\, (2\, xd\, xs\, yd + xd^2\, ys + 3\, xs^2\, ys)$

```
Simplify[
 IExy2 /. {x0 -> xs - xd, x1 -> xs + xd, y0 -> ys - yd, y1 -> ys + yd}]
```

$-\frac{1}{3}\, xd\, (2\, xd\, yd\, ys + xs\, (yd^2 + 3\, ys^2))$

```
Simplify[
 IEyy2 /. {x0 -> xs - xd, x1 -> xs + xd, y0 -> ys - yd, y1 -> ys + yd}]
```

$\frac{2}{3}\, yd\, (2\, xd\, yd\, ys + xs\, (yd^2 + 3\, ys^2))$

```
Simplify[
 IExy3 /. {x0 -> xs - xd, x1 -> xs + xd, y0 -> ys - yd, y1 -> ys + yd}]
```

$\frac{1}{3}\, yd\, (2\, xd\, xs\, yd + xd^2\, ys + 3\, xs^2\, ys)$